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VARIANTS OF KAZHDAN'S PROPERTY FOR SUBGROUPS OF SEMISIMPLE GROUPS

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ABSTRACT

Some variants of Kazhdan's property (T) for discrete groups are presented. It is shown that some groups (e.g. $SL_n(\mathbf{Q})$, $n \geqq 3$) which do not have property (T) still have some of these weaker properties. Applications to cohomology and infinitesimal rigidity for certain actions on manifolds are derived.

In this paper we discuss some variants of Kazhdan's property (T) for discrete groups. There are four sections in this paper. The first presents a discussion of some of the variants of property T with which we shall be dealing. The second contains a proof that lattices in certain locally compact groups which are products of algebraic groups over local fields, and the set of rational points of a suitable group over a global field have one of these basic variants of Kazhdan's property even though they do not necessarily possess the latter property. Section three contains some applications to group cohomology. The final section contains an application to infinitesimal rigidity for certain actions on manifolds, a result which is new even for groups with property T.

1. Variants of Kazhdan's property

Let G be a locally compact group, Rep(G) the equivalence classes of unitary representations, $G^\wedge$ the unitary dual, i.e., the set of equivalence classes of

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irreducible representations in Rep(G). $G^\wedge$ is equipped with the standard unitary dual topology. $I \in G^\wedge$ denotes the trivial representation. We recall that G is said to have Kazhdan's property (T) if I is an isolated point in $G^\wedge$.

Definition 1.1. If $R \subset G^\wedge$, we say that G has property $(T; R)$ if I is isolated in $R \cup \{I\}$ with the relative topology.

Thus, Kazhdan's property is property $(T; G^\wedge)$. Kazhdan proved that any simple algebraic group of split rank at least 2 over a local field has this property, as does any lattice in such a group. There are some natural and interesting examples of groups not satisfying $(T; G^\wedge)$ but satisfying $(T; R)$ for some natural subclasses R. Namely, let Γ be a discrete group, and F the set of $\rho \in \Gamma^\wedge$ which factor through a finite quotient of Γ. The condition that Γ satisfies $(T; F)$ turns out to be equivalent to some combinatorial and geometric properties of Γ. Before stating this, we first present some notation.

Let $X = (V, E)$ be a finite connected graph with a set of vertices V and edges E. Let d denote the canonical distance function on X and Δ_X (or simply Δ when there is no confusion) be the map $\Delta : L^2(X) \to L^2(X)$ given by

$$(\Delta f)(x) = (\deg(x))^{-1} \sum_{y \in V} (f(x) - f(y)),$$

where $\deg(x) = \operatorname{card}\{ y \mid d(x, y) = 1\}$.

Then Δ is a self-adjoint operator with non-negative eigenvalues. $\lambda_0 = 0$ is always an eigenvalue with the constants as eigenfunctions, and since X is connected, it is an eigenvalue of multiplicity 1. Let $\lambda_1(X)$ denote the next smallest eigenvalue. The graph X will be called a c-expander if for every subset $A \subset V$ for which $|A| \leqq |V|/2$, we have $|\partial A| \geqq c|A|$, where $\partial A = \{y \in V \mid d(y, A) = 1\}$.

Now let M be a compact Riemannian manifold, Δ the Laplace–Beltrami operator, and $\lambda_1(M)$ its smallest non-zero eigenvalue. The isoperimetric invariant of M is defined to be $h(M) = \inf\{\operatorname{area}(H)/\min(\operatorname{vol}(A), \operatorname{vol}(B)) \mid H$ is a closed hypersurface in M which divides M into two connected components A and $B\}$. We can now state:

Proposition 1.2. *Let Γ be a finitely generated group with a finite generating set S. Let $\mathcal{S}$ be a family of finite index normal subgroups of Γ. Then the following conditions are equivalent:*

(i) *Γ has property $(T; R(\mathcal{S}))$, where $R(\mathcal{S})$ is the set of all irreducible representations of Γ that factor through a quotient with kernel in $\mathcal{S}$.*

(ii) *There is a constant* $c_2 > 0$ *such that all Cayley graphs* $X(\Gamma/N, S)$, *with* $N \in \mathcal{S}$, *are* c_2*-expanders.*

(iii) *There is a constant* $c_3 > 0$ *such that* $\lambda_1(X(\Gamma/N, S)) \geqq c_3$ *for every* $N \in \mathcal{S}$.

If in addition Γ *is the fundamental group of a compact Riemannian manifold* M, *and for every* $N \in \mathcal{S}$ *we denote by* M_N *the corresponding finite sheeted covering, we then also have the equivalence with the following properties:*

(iv) *There is* $c_4 > 0$ *such that* $h(M_N) \geqq c_4$ *for every* $N \in \mathcal{S}$.

(v) *There is* $c_5 > 0$ *such that* $\lambda_1(M_N) \geqq c_5$ *for every* $N \in \mathcal{S}$.

The proof of this proposition is not difficult and is contained essentially in [B1], [B2], [AM], [Al]. One can also work out the connections between the various constants.

A non-trivial example is the group $\Gamma = SL(2, \mathbf{Z})$ where $\mathcal{S}$ is the set of congruence subgroups of Γ; i.e.,

$$\mathcal{S} = \{\Gamma(m) = \ker(SL(2, \mathbf{Z}) \to SL(2, \mathbf{Z}/m\mathbf{Z}))\}.$$

By a theorem of Selberg, $\lambda_1(\mathbf{H}^2/\Gamma(m)) \geqq 3/16$, where $\mathbf{H}^2$ is the upper half plane, and hence Γ has property $(T; R(\mathcal{S}))$. (We remark that the conclusions of Proposition 1.2 hold for $SL(2, \mathbf{Z})$ and $\mathcal{S}$ even though $SL(2, \mathbf{Z})$ is not co-compact. See [B2].) On the other hand it is not difficult to see that $SL(2, \mathbf{Z})$ does not have property $(T; R(\mathcal{F}))$, where $\mathcal{F}$ denotes the set of all normal subgroups of finite index.

Another interesting example is $\Gamma = SL(2, \mathbf{Z}[1/p])$ where p is any prime. This group has property $(T; R(\mathcal{F}))$, but does not have property T. The latter assertion follows immediately from the fact that Γ is a dense subgroup of $SL(2, \mathbf{R})$. To see the first assertion, recall that Γ has an affirmative solution to the congruence subgroup problem [Se]. Thus, every finite quotient of Γ is a congruence quotient of $SL(2, \mathbf{Z})$. Selberg's theorem and Proposition 1.2 now apply to show that Γ has property $(T; R(\mathcal{F}))$. In this example Γ is an irreducible lattice in $SL(2, \mathbf{R}) \times SL(2, \mathbf{Q}_p)$. An example of a similar nature is $\Gamma = SL(2, \mathcal{O})$ where $\mathcal{O}$ is the ring of integers in a quadratic real number field. See [Sa]. Here Γ is a lattice in $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$. These two examples also have property $(T; \mathrm{FD})$, where FD denotes the space of finite dimensional unitary representations. In the examples this follows simply from the fact that every finite dimensional unitary representation of these groups has finite image. It would be interesting to determine whether or not properties $(T; \mathrm{FD})$ and $(T; R(\mathcal{F}))$ are equivalent in general, say for finitely generated residually finite groups.

We believe that any irreducible lattice in a non-trivial product (i.e., at least two non-compact factors) of semisimple groups (over any local fields) has property $(T; R(\mathscr{F}))$. A main result of the next section proves this under the hypothesis that one of the factors of the product has property (T). In particular, for real groups, the only remaining cases are products of groups of the form $\mathrm{SO}(1, n)$ and $\mathrm{SU}(1, n)$. This conjecture in fact follows from two other well known conjectures. First, we recall that Serre has conjectured that the congruence subgroup property holds for any irreducible arithmetic lattice in a non-trivial product of semisimple groups. Second, it is a general conjecture that if G is any semisimple group and Γ is any arithmetic lattice, then there is $c > 0$ such that $\lambda_1(M_N) \geqq c$ for every $N \in \mathscr{S}$ where $\mathscr{S}$ is the set of congruence subgroups of Γ and Γ is viewed as the fundamental group of the corresponding locally symmetric space. Thus, our conjecture would follow from the validity of these two conjectures (using Margulis arithmeticity theorem as well.) As to the second of these conjectures, as in the previous paragraph, the problem is reduced to the case of $\mathrm{SO}(1, n)$, $\mathrm{SU}(1, n)$, and products of these groups. For $\mathrm{SO}(1, n)$, it has recently been verified in [EGM] and [LPS], and for $\mathrm{SU}(1, n)$ in [Li].

Property $(T, R(\mathscr{F}))$ also arises naturally in studying fundamental groups of spaces on which a semisimple Lie group can act. See [Z2].

Another natural class of unitary representations are those whose matrix coefficients vanish at ∞ (or "mixing" representations.) We denote the elements of $G^\wedge$ having this property by R_∞. In the next section we deduce that the irreducible lattices we are considering will have property (T, R_∞) even though they may not have property T. The arguments will apply to give a new type of result for groups over global fields. For example, we show in the next section that $\mathrm{SL}(n, \mathbf{Q})$ has property (T, R_∞), even though it is not finitely generated and hence does not have property T. This result holds for other $\mathbf{Q}$-groups, and via the operation of restriction of scalars (see [Z1]), similar results can be obtained for groups defined over algebraic number fields.

2. Irreducible lattices in a product of groups

To ease our notation, by a semisimple group we shall mean a group of the form ΠH_i, where the product is finite, and each H_i is the group of k_i-points of a semisimple algebraic k_i-group, where k_i is a local field. In this section we show that irreducible lattices in some such products, even without property T, will have a relative version of property T. This is sufficient to deduce, for example, that Γ has a finite abelianization, and in certain cases that Γ is finitely generated. We shall see other applications as well. We also obtain results for lattices in semisimple groups over the adeles.

DEFINITION 2.1. Suppose $h: G \to H$ is a homomorphism of locally compact groups. If $\pi \in \mathrm{Rep}(G)$, we say that π degenerates through h (or through H) if for some $k \geqq 1$, some non-trivial (i.e., not a multiple of I) subrepresentation of $S^k(\pi)$ (the k-fold symmetric power of π) factors through h.

THEOREM 2.2. *Suppose for $i = 1, 2$, G_i is a locally compact separable group, and that G_2 is Kazhdan. Let $\Gamma \subset G = G_1 \times G_2$ be a lattice which projects densely onto G_1, and let $p_i : \Gamma \to G_i$ be the projection. Let*

$$R_1 = \{\pi \in \Gamma^{\wedge} \mid \pi \text{ does not degenerate through } p_1\}.$$

Then Γ has property $(T; R_1)$.

For the proof we need the following lemma.

LEMMA 2.3. *Suppose Γ is as in the statement of Theorem 2.2 and suppose that Γ acts ergodically in a finite measure preserving way on a space X. Assume that the representation of Γ on $L^2(X)_0$ $(= \mathbf{C}^{\perp} \subset L^2(X))$ weakly contains I. Then there is a finite measure preserving G_1-action on a space Y and a measure preserving Γ-map $X \to Y$.*

PROOF. Let Z be the G-space induced from X, i.e., $Z = (X \times G)/\Gamma$. Let π be the representation of Γ on $L^2(X)_0$, and σ the representation of G induced from π. Then we have a natural identification of σ as a subrepresentation of $L^2(Z)_0$. Since π weakly contains I, so does σ. Since G_2 is Kazhdan, there are G_2 invariant vectors in $L^2(Z)_0$, i.e., G_2 does not act ergodically on Z. (We remark that G does act ergodically on Z, since the Γ action on X is ergodic.) Let E be the space of G_2-ergodic components of the G_2 action on Z. Then G acts on E and we have a G-map $Z \to E$, with G_2 acting trivially on E. We have a natural Γ-embedding $X \to Z$. The image is of measure 0, but its saturation under G is of full measure. The image carries a finite Γ-invariant measure, and hence E carries a finite Γ-invariant measure as well. Since we may assume (by a result of Varadarajan; see [Z1, 2.1.19]) that E is a compact metric G_1-space, this Γ-invariant measure will be G_1 invariant as well, since $p_1(\Gamma)$ is dense in G_1. Thus $X \to E$ is the required map.

REMARK 2.4. The proof actually shows a more precise statement which we shall need later. We recall that if π is a unitary representation of a group G, $K \subset G$ is a subset, $\varepsilon > 0$, and v is a unit vector in the representation space, then

v is called (ε, K)-invariant if $\| \pi(g)v - v \| < \varepsilon$ for all $g \in K$. If G is generated by K, then Kazhdan's property is equivalent to the existence of some $\varepsilon > 0$ such that the existence of (ε, K)-invariant vectors for π implies that π has non-trivial invariant vectors. The proof of Lemma 2.3 combined with the argument of [Z1, 9.1.1] yields the following assertion, assuming the hypotheses of Lemma 2.3. There is a finite set $S \subset \Gamma$, and some $\varepsilon > 0$ such that any ergodic Γ-space X with invariant probability measure and some $f \in L^2(X)_0$ which is (ε, S)-invariant satisfies the conclusion of Lemma 2.3. If Γ is finitely generated, then for any finite generating set S we may find such an ε.

Proof of Theorem 2.2. Let R_0 be the subset of finite dimensional representations in R_1, and suppose that I is not isolated in R_0. Let π_i be a sequence in R_0 with $\pi_i \to I$. Let π be the direct sum of the π_i, and K the closure of the image of π in the direct product of the corresponding unitary groups. Thus we have a homomorphism $h : \Gamma \to K$ with dense image such that the representation of Γ on $L^2(K)_0$ weakly contains the the identity. On the other hand, if I is isolated in R_0 but not in R_1, let π be an infinite dimensional irreducible representation with an (ε, S)-invariant unit vector, where ε and S are as in the remark above. Let X be the Gaussian Γ-space associated to this unitary representation [Z1]. Then Γ acts ergodically on X and we can identify $L^2(X)$ naturally with $\Sigma_{k \geq 0}^{\oplus} S^k(\pi)$. Thus, in either case we have an ergodic Γ-space X with finite invariant measure such that $L^2(X)_0$ contains (ε, S)-invariant vectors and has no subrepresentations that factor through G_1. However, if we take Y as in Lemma 2.3, we have $L^2(Y)_0 \subset L^2(X)_0$ as unitary Γ-modules, which is a contradiction.

Remark 2.5. We have used here the following simple fact. If $V \subset W$ are unitary Γ-modules, and V factors through G_1, and $L \subset W$ is Γ-invariant with orthogonal projection $P : W \to L$, then $P(V)$ factors through G_1. This follows easily from the remark that the representation of Γ on this space is (uniformly) continuous where Γ has the topology of a subgroup of G_1.

Corollary 2.6. (i) *Let Γ be as in Theorem 2.2, and assume G_1 is minimally almost periodic (i.e., has no non-trivial finite dimensional unitary representations.) Then Γ has property $(T; \mathrm{FD})$, and a fortiori has property $(T; R(\mathscr{F}))$. In particular, this holds for irreducible lattices in semisimple groups in which at least one of the factors is non-compact and Kazhdan.*

(ii) *Let Γ be an irreducible lattice in a semisimple real Lie group in which at least one of the factors is of real rank $\geqq 2$. Let $X = G/K$ be the symmetric space* *associated with G. Then there is some $\varepsilon > 0$ such that $\lambda_1(X/\Lambda) > \varepsilon$ for every finite index subgroup $\Lambda \subset \Gamma$.*

PROOF. (i) implies (ii) by Proposition 1.2. To prove (i), simply observe that if π is finite dimensional, so is $S^k(\pi)$, and thus any subrepresentation of $S^k(\pi)$ extending to G_1 via p_1 is trivial.

COROLLARY 2.7. *Suppose Γ is a discrete group with property (T; FD) (e.g., let Γ be as in Theorem 2.2.) Then Γ^{ab} is finite.*

PROOF. If not, the dual group of Γ^{ab} is non-discrete.

We now consider the issue of finite generation. Unlike the case of property T, it is not true that property (T; FD) implies finite generation, even for residually finite groups. For example, let P be the set of prime numbers except for one prime, say p. Let $\mathcal{O}$ be the ring of P-integers, and let $\Gamma = \mathrm{SL}(n, \mathcal{O})$. Since P is infinite, Γ is not finitely generated. It is however residually finite, since every element will have non-trivial image when reduced modulo p^r for a sufficiently large r. To see property (T; FD), we observe that by the congruence subgroup property the profinite completion of Γ is the same as the p-adic completion of $\mathrm{SL}(n, \mathbf{Z})$. We note that in Proposition 1.2 we have (ii) implies (i) holds as long as S generates all finite quotients of Γ even if S does not generate Γ. Therefore property (T; FD) for Γ follows form Selberg's theorem for $n = 2$, or property T for $n > 2$.

On the other hand, consider once again the situation in Theorem 2.2. By a result of Wang [W], if H is a connected semisimple real Lie group, then any dense subgroup contains a finitely generated dense subgroup. With this additional condition, we can deduce finite generation.

PROPOSITION 2.8. *Assume the hypotheses of Theorem 2.2, and that* $\ker(p_1)$ *is finite. Suppose further that G_1 satisfies the condition that any dense subgroup contains a finitely generated dense subgroup. Then Γ is finitely generated.*

PROOF. We can write $\Gamma = \bigcup \Gamma_n$, where each Γ_n is a finitely generated subgroup which projects densely into G_1. Then the representation of Γ on the direct sum of $L^2(\Gamma/\Gamma_n)$ weakly contains I, and hence so does this representation induced to G. It follows that G_2 has a non-trivial invariant function f in some $L^2(G/\Gamma_n)$. Since Γ_n is dense upon projection to G_1, $G_2\Gamma_n$ is dense in G. It follows that f is essentially constant, and hence that Γ_n is a lattice in G. Thus, Γ_n is of finite index in Γ, and since Γ_n is finitely generated, so is Γ.

Let R_∞ be the set of irreducible representations whose matrix coefficients vanish at ∞ (or "mixing" representations.)

COROLLARY 2.9. *Let Γ be as in Theorem 2.2. Then Γ has property $(T; R_\infty)$.*

PROOF. Since π is mixing, so is $S^k(\pi)$. Thus as $a_n \in \Gamma$ goes to ∞, $S^k(\pi)(a_n) \to 0$ in the weak operator topology. However, we can choose $a_n \to \infty$ in Γ but $a_n \to e$ in G_1 since Γ projects densely into G_1. Hence, no subrepresentation can factor through G_1.

EXAMPLE 2.10. Let G be an algebraic $\mathbf{Q}$-group such that $G(\mathbf{R})$ is a simple non-compact Lie group with Kazhdan's property. Then $G(\mathbf{Q})$ has property $(T; R_\infty)$. To see this, let $\mathscr{A}$ be the ring of adeles, and $\mathscr{A}_f$ the finite adeles. Then $G(\mathbf{Q})$ is an irreducible lattice in $G(\mathscr{A})$, where the latter is written as $G(\mathbf{R}) \times G(\mathscr{A}_f)$. The assertion then follows from 2.9. The argument can clearly be made to apply to more general groups over number fields.

3. Applications to group cohomology

In this section we prove some vanishing theorems for certain first cohomology groups, which are already known for groups with property T. In the case of semisimple groups, the first result follows from results in [BW]. However, our argument is much more elementary.

THEOREM 3.1. *Let Γ be as in Theorem 2.2, with G_1 minimally almost periodic. If V is a finite dimensional unitary Γ-module, then $H^1(\Gamma; V) = 0$.*

To prove Theorem 3.1, we first make some ergodic theoretic observations of independent interest. By a measurably isometric action we mean one measurably conjugate to an ergodic isometric action on a compact metric space. We also use the notion of actions with generalized discrete spectrum (hereafter g.d.s.) [Z4], which we also take to be ergodic, and which is a natural generalization of the the notion of measurably isometric action. We shall be concerned with the first cohomology of Γ with coefficients in the collection of measurable functions (modulo null sets) on a measurable Γ-space X with values in a locally compact group L. This can be identified with the standard ergodic theoretic cohomology $H^1(X \times \Gamma, L)$. See [Z1], for example, for a discussion of this set. We recall here that if Γ has property T, X has a finite Γ-invariant measure, and L is an amenable group with no non-trivial compact subgroups, then $H^1(X \times \Gamma, L) = 0$ [Z1, 9.1.1]. This result is extremely useful in geometric applications. See [Z5], for example. Here, we shall establish this type of vanishing for Γ as in Theorem 2.2, for a restricted class of X. This will then be sufficient to give a proof of Theorem 3.1.

LEMMA 3.2. *Let Γ be as in Theorem 3.1, and let X be a Γ-space with g.d.s. Then X has no non-trivial quotient action which factors through G_1.*

PROOF. If Y is such a quotient, Y has g.d.s. [Z4], and hence Y has a measurably isometric quotient. This is impossible.

PROPOSITION 3.3. *Let X be as in Lemma 3.2. If $\alpha_i \in Z^1(X \times \Gamma, S^1)$, and $\alpha_i \to I$ in the sense of [Z6], then some α_i is trivial in cohomology.*

PROOF. Suppose not. Let $\alpha = (\alpha_1, \alpha_2, \ldots) : X \times \Gamma \to \Pi S^1$, and let K be the Mackey range [Z3]. Then K is a compact abelian group, and (possibly after replacing α with a cohomologous cocycle), we have $\alpha : X \times \Gamma \to K$ with Mackey dense range, and elements $\lambda_i \in K^\wedge$ such that $\lambda_i \circ \alpha \to I$. Form the skew product $Z = X \times_\alpha K$. Then $L^2(Z) \supset \Sigma^\oplus L^2(X)_i$, where Γ acts on $L^2(X)_i$ via translation and a twist by $\lambda_i \circ \alpha$. It follows that $L^2(Z)_0$ weakly contains I, and hence, by Lemma 2.3, Z has a non-trivial quotient Γ-space that factors through G_1. Since Z has g.d.s., this is impossible by Lemma 3.2.

PROPOSITION 3.4. *If Γ acts on X with g.d.s. (and Γ as above), then $H^1(X \times \Gamma, \mathbf{R}) = 0$ (and hence $H^1(X \times \Gamma, \mathbf{Z}) = 0$).*

PROOF. By results of Moore–Schmidt [MS], it suffices to see for a fixed cocycle α, that $\lambda \circ \alpha$ is trivial for all $\lambda \in \mathbf{R}^\wedge$. If $\{\lambda \in \mathbf{R}^\wedge \mid \lambda \circ \alpha \text{ is trivial}\}$ contains an open neighborhood of $I \in \mathbf{R}^\wedge$, we are done. If not, choose $\lambda_i \in \mathbf{R}^\wedge$ such that $\lambda_i \to I$ and $\lambda_i \circ \alpha$ non-trivial. But $\lambda_i \to I$ implies $\lambda_i \circ \alpha \to I$, contradicting Proposition 3.3.

COROLLARY 3.5. *Let Γ be as in Theorem 3.1, and let X be a Γ-space with g.d.s. Then $H^1(X \times \Gamma, U) = 0$ for any unipotent Lie group or finitely generated torsion free nilpotent group.*

PROOF OF THEOREM 3.1. We can assume we have a homomorphism $h : \Gamma \to H = K \times_S V$ (semidirect product) where V is a Euclidean group and K is a compact group, such that the image of Γ in K is dense. We wish to show that the closure of $h(\Gamma)$ is compact. The map $H \to K$ of Γ spaces defines an element of $H^1(K \times \Gamma, V)$ ~~(viewing K as an isometric Γ-space.) By Corollary~~ 3.5 this cocycle is trivial in cohomology, which implies that there is a measurable Γ-invariant section $K \to H$. This in turn implies that there is a finite Γ-

invariant measure on H. This measure must be $h(\Gamma)^-$-invariant, and since $h(\Gamma)^-$ acts properly on H, $h(\Gamma)^-$ must be compact.

4. Application to infinitesimal rigidity

If M is a compact manifold and Γ is a discrete group acting smoothly on M, the action is called infinitesimally rigid if $H^1(\Gamma, V(M)) = 0$, where $V(M)$ is the Γ-module of smooth vector fields on M. For lattices in semisimple groups, a number of natural actions on homogeneous spaces were shown to satisfy this property in [Z7]. Here we establish (much more easily) infinitesimal rigidity of ergodic isometric actions for a class of groups which includes all those of Theorem 3.1, and all those with property T. The result is new even for groups with property T.

THEOREM 4.1. *Suppose Γ is a finitely generated group satisfying:*
(a) *Γ has property (T, FD); and*
(b) *$H^1(\Gamma; V) = 0$ for all finite dimensional unitary Γ-modules.*
Suppose Γ acts isometrically and ergodically on a compact manifold M. Then the Γ action on M is infinitesimally rigid. Furthermore, $H^1(\Gamma, C^\infty(M)) = 0$.

REMARK 4.2. Property T implies both (a) and (b). For Γ as in Theorem 3.1, both are satisfied as well by Corollary 2.6 and Theorem 3.1.

PROOF. Let $V^2(M)$ be the space of L^2 vector fields, $V^{2,k}(M)$ the $(2, k)$-Sobolev space of vector fields. If Γ acts on M, these are all Γ modules in a natural way. Let K be the isometry group of M, so that $M = K/K_0$ for some closed subgroup K_0, by ergodicity of Γ on M. We have a homomorphism $\Gamma \to K$ defining the action on M. $V^2(M)$ is a unitary K-module, and we can write $V^2(M) = \Sigma^{\oplus} V_i$, where $V_i \subset V(M)$, $\dim V_i < \infty$, and V_i are mutually disjoint K-modules. Let Δ be the Laplace operator on $V(M)$. Then Δ commutes with K, and hence $\Delta(V_i) \subset V_i$ for all i. Further, $(I + \Delta^k)^{1/2} : V^{2,k}(M) \to V^2(M)$ is an isomorphism. Thus, if $h = \Sigma^{\oplus} h_i \in V^2(M)$, where $h_i \in V_i$ (and in particular $\Sigma \| h_i \|^2 < \infty$), we have $h \in V(M)$ if and only if $\Sigma \| \Delta^k h_i \|^2 < \infty$ for all k.

Now suppose $f : \Gamma \to V(M)$ is a 1-cocycle. Then we can write $f(a) = \Sigma f_i(a)$, where $f_i : \Gamma \to V_i$ is a 1-cocycle. Thus, we can find $h_i \in V_i$ such that for all $a \in \Gamma$ we have $f_i(a) = ah_i - h_i$. Fix a finite generating set $F \subset \Gamma$. By property $(T; \mathrm{FD})$, there is $\varepsilon > 0$ such that for each i we can find $a_i \in F$ such that $\| a_i h_i - h_i \| \geqq \varepsilon \| h_i \|$. Thus, $\| h_i \| \leqq \varepsilon^{-1} \| f_i(a_i) \|$ for some $a_i \in F$. Hence $\Sigma \| h_i \|^2 \leqq \varepsilon^{-2} \Sigma_{a \in F} \| f(a) \|^2 < \infty$. Therefore $h = \Sigma h_i$ defines an element of $V^2(M)$ with $f(a) = ah - h$ for all $a \in \Gamma$. It suffices to show that

$h \in V(M)$, and hence by the remarks above that $\Sigma \| \Delta^k h_i \|^2 < \infty$ for all k. However, since Δ commutes with Γ, we have $\Delta^k f \in Z^1(\Gamma, V(M))$, $\Delta^k f = \Sigma \Delta^k f_i$, where $\Delta^k f_i \in V_i$, and $\Delta^k f_i(a) = a\Delta^k h_i - \Delta^k h_i$. Repeating the above argument for $\Delta^k h_i$ yields the required assertion.

The argument for the module $C^\infty(M)$ is similar.

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